5a. Development of the Conservation of Mass Equation

Energy and mass are fundamental components of a physical system. The ability to understand and mathematically model both components is basic to any engineering or scientific application. Mass is always conserved. It cannot be created nor destroyed by ordinary means. And the interrelationship of mass and energy on an atomic level ($E = mc^2$) is commonplace to almost everyone. In classical thermodynamics it is assumed that energy and mass are separate entities; this Einsteinium interrelationship is left to the physicists and advanced studies in statistical thermodynamics.

5a.1. The Continuity Equation

In thermodynamic systems, the crux of classical study, mass crossing the system boundary is mathematically treated through infinitesimal analysis with both time and location as independent variables. Such analysis will yield the traditional conservation of mass relationship.

Consider the following system of arbitrary design:

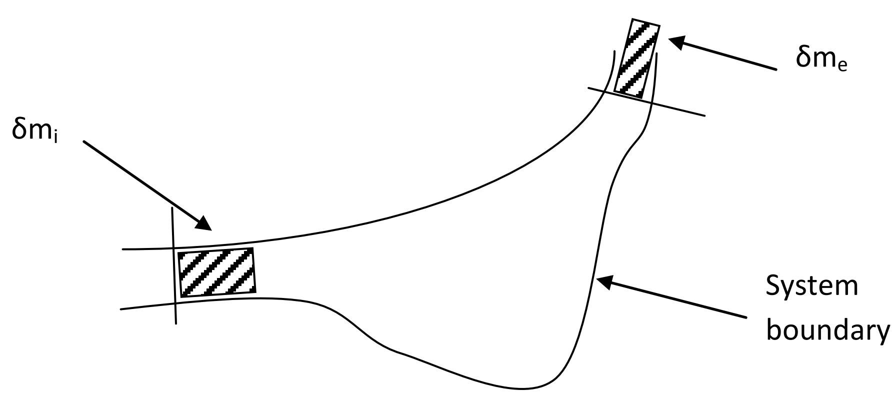

Figure 5a.1 Schematic of a system for analysis of the continuity equation.

At time t the mass within the system is designated as m_t. During an interval in time δt the mass entering the system is δm_i and the mass leaving the system is δm_e. Then from the conservation of mass (mass is always conserved) a mathematical representation is

$$m_t + \delta m_i - \delta m_e = m_{t+\delta t}$$

Rearranging

$$m_{t+\delta t} - m_t = \delta m_e - \delta m_i$$

Expressing this as a rate equation by dividing by an infinitesimal step in time δt yields

$$\frac{m_{t+\delta t} - m_t}{\delta t} = \frac{\delta m_e}{\delta t} - \frac{\delta m_i}{\delta t}$$

And taking the limit as $\delta t \rightarrow 0$ with the following symbolic definitions

$$\lim_{t \rightarrow 0} \frac{m_{t+\delta t} - m_t}{\delta t} = \left. \frac{dm}{dt} \right|_{system}$$

$$\lim_{t \rightarrow 0} \frac{\delta m_e}{\delta t} = \dot{m}_e$$

$$\lim_{t \rightarrow 0} \frac{\delta m_i}{\delta t} = \dot{m}_i$$

In the above it is essential to understand that the quantity $\left. \frac{dm}{dt} \right|_{system}$ is the rate of change of mass within the system and $\dot{m}_e$ and $\dot{m}_i$ are the rates of mass crossing the boundary of the system (exiting and entering the system).

Thus

$$\left. \frac{dm}{dt} \right|_{system} = \dot{m}_e - \dot{m}_i$$

Since mass may cross the system boundary in several areas (such as in mixing processes, etc.) the above equation better represents all encounters of conservation of mass

$$\left. \frac{dm}{dt} \right|_{system} = \Sigma \dot{m}_e - \Sigma \dot{m}_i$$

In engineering and scientific jargon the above conservation of mass equation is frequently referred to as the continuity equation.

5a.2. Mass Flow Rates

This form of the continuity equation is sufficient for the majority of applications. The above represents average fluid conditions but since the matter inside the system or crossing the system boundary may not be uniform at any given instant in time it is often convenient to express each of these mass quantities in terms of local thermodynamic fluid properties.

To address the issue of non-uniformity within the system, the matter within the system may be divided into a subset of arbitrarily small elements. The total mass inside of the system m_{system} at any instant of time is

$$m_{system} = \int_V \rho dV \Big|_{system}$$

Where ρ is the density of the matter within the elemental volume dV and the integral of all dV elements is the total volume V

The rate of change of mass within the system is expressed

$$\frac{dm}{dt}\Big|_{system} = \frac{d}{dt}\int_V \rho dV \Big|_{ststem}$$

The mass flow rate crossing the system boundary (both entering and leaving) is more difficult to represent since the mass flow crossing the boundary is most likely not uniform or normal to the surface, and the boundary may be moving. Consider the following

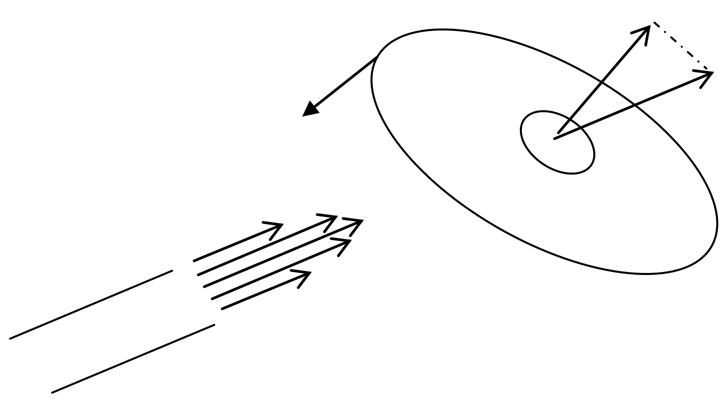

Figure 5a.2 Non-uniform and oblique flow across a moving system boundary.

$$\delta \dot{m} = \rho V_{rn} \delta A$$

$$\dot{m} = \int_A \rho V_{rn} \delta A$$

$$\frac{d}{dt} \int_V \rho dV + \int_A \rho V_{rn} \delta A = 0$$

Using a parallel development to the simplified continuity equation developed above and repeated here for clarity

$$\left.\frac{dm}{dt}\right|_{system} = \Sigma \dot{m}_e - \Sigma \dot{m}_i$$

$$\left.\frac{d}{dt} \int_V \rho dV\right|_{system} = \left.\int_A \rho V_{rn} \delta A\right|_{exiting} - \left.\int_A \rho V_{rn} \delta A\right|_{entering}$$

This form of the continuity equation is basic to subsequent development in the fields of heat transfer and fluid mechanics which are concerned with state and velocity variations. In classical thermodynamics at this elementary level bulk average values for local thermodynamic values are used to simplify the solution (often with very good accuracy) and the simplified continuity equation is sufficient. Further, if the following assumptions are made (again, with very good accuracy):

1. The control surface is stationary
2. The flow is normal to the boundary surface
3. The thermodynamics state and fluid velocity are uniform (bulk average values over the flow area at any instant in time.

Then the mass flow rate crossing the system boundary (either exiting or entering) can be expressed simply as

$$\dot{m} = \int_A \rho V_{rn} \delta A = \rho A V$$

Or simply

$$\dot{m} = \rho A V$$

When applying the above equation the significance of each of the assumptions should be understood. The conservation of mass equation (along with these assumptions) is inherent in the conservation of energy equation, the First Law of Thermodynamics, which follows.